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DIPARTIMENTO DI ECONOMIA, ISTITUZIONI, TERRITORIO

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Corrado Benassi – Alessandra Chirco – Caterina Colombo

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Corrado Benassi
Dipartimento di Scienze Economiche
Alma Mater Studiorum - Università di Bologna

Alessandra Chirco
Dipartimento di Scienze Economiche e Matematico-Statistiche
Università di Lecce

Caterina Colombo
Dipartimento di Economia Istituzioni Territorio
Università di Ferrara

Abstract We present a duopoly model with vertical product differentiation and uncovered market, where the distribution of the consumers' willingness to pay.is non-uniform. By using a trapezoid distribution we solve explicitly for market equilibrium as a function of a mean preserving spread of the income distribution. We find that the relationship between the latter and all relevant market variables is typically non monotone, i.e. strongly dependent on initial conditions. Indeed, these influence the interplay of the two forces affecting each firm's demand and its elasticity: greater income concentration makes for new consumers entering the market and redistributes demand between firms.

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Keywords Vertical differentiation, Income concentration, Mean preserving spread.

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Corresponding author Corrado Benassi, Dipartimento di Scienze Economiche, Università di Bologna, Piazza Scaravilli 2, 40126 Bologna, Italy. Phone: +39 051 2098667; Fax: +39 051 221968;
e-mail: benassi@economia.unibo.it

1 Introduzione

In the analysis of duopolistic markets with vertical product differentiation, a key question is how market equilibrium is affected by different degrees of income heterogeneity among consumers. Answers to this question have been provided in a twofold perspective. On the one hand, the assumption of covered market has been invoked to extend the generalization of the standard horizontal differentiation model to the vertical differentiation case (e.g., Anderson *et al.*, 1997). On the other hand, the assumption of uniform distribution of the consumers' willingness to pay has been used to study how the solution of the model depends on the degree of *relative* heterogeneity, measured by the ratio between the endpoint of the support of the income distribution – which showed that the market may or not be covered at equilibrum, depending on the length of the support as well as its position on the real line (Wauthy, 1996): in this framework, the only distributional shocks whose effects have been explicitly modeled are a "stretching" of the support of a uniform distribution, or its horizontal shift along the real line (Gabszewicz and Thisse, 1979).

This paper studies the equilibrium configuration of the standard duopoly version of the Mussa - Rosen (1978) model with uncovered market and costless quality choice, under the assumption that the consumers' income distribution is altered by variations in the degree of dispersion for a given support – we use a mean preserving spread to model income concentration, higher levels of which modify a symmetric trapezoid distribution of the willingness to pay, starting from the standard case of the uniform distribution to the limit case of the triangular one.

Our main result is that, unlike what happens in horizontal differentiation models, the effects of a higher concentration of consumers on the firms' optimal choices depend (both quantitatively and qualitatively) on the degree of concentration one starts with. In other word, a given increase in the frequencies of the central percentiles has very different implications, according as it affects a very dispersed, or a relatively concentrated, initial distribution. Very generally speaking, two basic mechanisms are at work as income gets more concentrated around its mean: first, new classes of consumers enter the market, who used not to buy at all before the distributional shock; second, the willingness to pay is increased, of those consumers who did buy before the distributional shock. If the latter hits a highly dispersed distribution, the first effect dominates, while the second effect is stronger when dispersion is low at the inital conditions. Obviously, the relative weight of these two effects is the key factor affecting the firms' optimal reactions to the given shock, both in terms of prices and of optimum quality. Given that the high quality level is standardly fixed at its highest value as quality increases are costless, neither the firms' market shares, nor their prices, nor the (low) quality level show a monotonic behavior as the degree of income concentration varies.

The paper is organized as follows. In section 2 we outline the general framework of our analysis by modelling market demand under a trapezoid distribution of the consumers' incomes. Section 3 offers the complete analytical solution for market equilibrium as a function of the degree of income concentration. A discussion of the economic mechanisms underlying this solution is provided in section 4. Some remarks about the robustness of our results and final comments are gathered in section 5.

2 The general framework

We consider the standard model of a market for a vertically differentiated product, where two competing firms, $i = H, L$, play a non-cooperative two-stage game on price and quality. Each firm i produces a good of quality $s_i \in [0, \overline{s}]$ at a cost independent of s_i, which we normalize to zero. The firm choosing the highest quality s_H sells its product at a price p_H, while goods of the lowest quality s_L are sold at a price p_L.

Consumers differ according to their taste for quality, θ. Each consumer j is characterized by the following utility function (Mussa and Rosen, 1978):

$$\begin{aligned} U_j &= \theta_j s - p \geq 0 \text{ if she purchases a unit of good of quality } s \\ U_j &= 0 \qquad\qquad\quad \text{if no purchase is made} \end{aligned}$$

where higher values of θ_j signal a higher willingness to pay for quality. We make the standard assumptions to the effect that $\theta_H = (p_H - p_L) / (s_H - s_L)$ identifies the marginal consumer who is indifferent between buying the high and the low quality goods, while $\theta_L = p_L / s_L$ identifies the marginal consumer indifferent between purchasing the low quality good and nothing at all.[1]

As is well known (e.g., Tirole, 1988, p.97), the preference parameter θ can also be interpreted as an income index, so that the distribution of θ across the population of consumers may be taken as a proxy for the distribution of their incomes. Actually, it is also well known that this interpretation of the consumers' heterogeneity raises some doubts about the generality of the

[1] Indeed, if $p_L/s_L < p_H/s_H$ (as is standardly verified in equilibrium), the above preferences imply that all consumers characterized by $\theta_j \geq p_L/s_L = \theta_L$ buy a unit of the good. The high quality version is purchased by consumers with $\theta_j \geq (p_H - p_L) / (s_H - s_L) = \theta_H$, while the low quality is bought by those with $\theta_L \leq \theta_j < \theta_H$.

standard results (e.g., Wang, 2003), which are obtained under the hypothesis of uniform distribution of θ – and indeed the latter is the assumption we want to do away with. In order to investigate how the degree of income concentration affects market equilibrium, we normalize the population to 1 and assume that the indicator θ is distributed across this population according to a continuous symmetric trapezoid density $f(\theta, u)$,[2] defined over the support $[0, 1]$; the parameter $u \in [0, 1]$ is the length of the shortest base, and is accordingly an inverse index of concentration.[3] In particular, the density $f(\theta, u)$ is defined as follows:

$$\text{for } u = 1 \qquad f(\theta, 1) = 1 \qquad \text{for } \theta \in [0, 1]$$

$$\text{for } u \in [0, 1) \qquad f(\theta, u) = \begin{cases} \dfrac{4}{1-u^2}\theta & \text{for } \theta \in A = \left[0, \dfrac{1-u}{2}\right] \\[2ex] \dfrac{2}{1+u} & \text{for } \theta \in B = \left[\dfrac{1-u}{2}, \dfrac{1+u}{2}\right] \\[2ex] \dfrac{4}{1-u^2}(1-\theta) & \text{for } \theta \in C = \left[\dfrac{1+u}{2}, 1\right) \end{cases}$$

Figure 1 (drawn by way of example under the values $u = 1/2$ and $u = 1/4$) gives an immediate interpretation of the intervals A, B and C. Interval B is the projection of the shortest base onto the support, and we shall call it the 'modal area'; A and C are the projections onto the support, respectively of the left and right oblique sides of the trapezoid – with some *abus de language*, we shall call them the 'left tail area' and the 'right tail area'.

[2] The same density has been used by Scrimitore (2005) in the framework of horzontal product differentiation.

[3] It can be checked that u is a mean preserving spread, which ranks equal-mean distributions by second-order stochastic dominance. It is well known that such ranking is equivalent to Lorenz dominance: u is thus an inequality index satisfying the Pigou-Dalton's "principle of transfers".

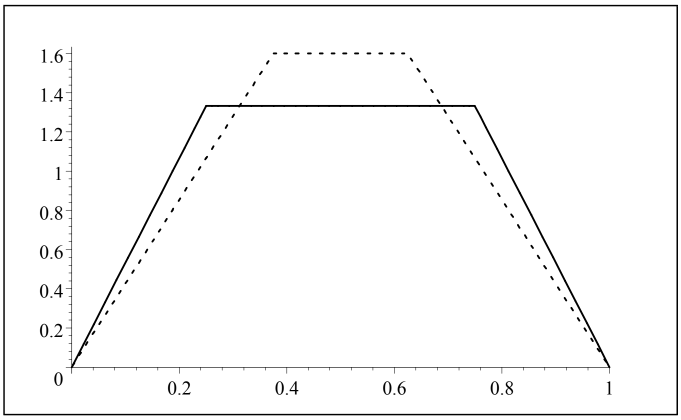

Figure 1: The densities $f(\theta, 1/2)$ (solid) and $f(\theta, 1/4)$ (dotted)

Given the above formulation, as u takes on lower and lower values, the 'income' distribution gets more concentrated around its mean value $\mu = 1/2$, from the highly dispersed rectangular distribution ($u = 1$) up to the most concentrated triangular distribution ($u = 0$). In order to trace the pattern of all relevant variables as a function of this measure of income concentration, we look for an explicit solution of the model parametrized on u.

First, we determine the demand faced by firms L and H, as a function of p_H, p_L, s_H and s_L:

$$D_H(\theta_H; u) = \int_{\theta_H}^{1} f(\theta, u)\, d\theta = 1 - F(\theta_H, u) \tag{1}$$

$$D_L(\theta_H, \theta_L; u) = \int_{\theta_L}^{\theta_H} f(\theta, u)\, d\theta = F(\theta_H, u) - F(\theta_L, u) \tag{2}$$

where $F(\theta, u) = \int_0^\theta f(z, u)dz$ is the distribution associated to the consumers' density $f(\cdot, u)$.

Clearly, the explicit formulations of (1) and (2) differ, according as the indifferent consumers θ_H and θ_L (i.e., the limits of integration in the above functions) are located in intervals A, B or C. In principle, six configurations are conceivable: (a) θ_L, $\theta_H \in B$; (b) θ_L, $\theta_H \in A$; (c) $\theta_L \in A$, $\theta_H \in B$; (d) $\theta_L \in B$, $\theta_H \in C$; (e) $\theta_L \in A$, $\theta_H \in C$; (f) θ_L, $\theta_H \in C$. However, the following proposition allows us to exclude the last three cases.

Proposition 1 *Consider a concave symmetric density $f(\theta)$ defined over $[0, 1]$, such that $f(0) = f(1) = 0$ and $f(1/2) \geq 1$. If (θ_H^*, θ_L^*), $\theta_H^* > \theta_L^*$, identify the marginal consumers at a perfect Nash Equilibrium in the two stage vertical differentiation game, then θ_H^* is lower than the median of the distribution.*

Proof See the Appendix

By Proposition 1, we can limit our analysis to cases (a) to (c) above, since in equilibrium no marginal consumer will ever be placed in the right tail area C. We shall now proceed as follows. First, we look for the solutions of the two-stage game in price and quality constrained by each of these three possible conjectures about the position of the indifferent consumers – which amounts in each case to constraining the strategies available to the firms, limiting their choices to those delivering the conjectured positions of the marginal consumers. Then we verify which of these three solutions actually holds in equilibrium, at the various possible values of the concentration parameter u.

3 The solution of the model

The optimal qualities and prices chosen by the two firms can be determined following the well known backward induction procedure, i.e. by solving first the price stage of the game.

3.1 Case (a): both marginal consumers in the modal area

If we limit ourselves to situations where $\theta_L, \theta_H \in B$, the demand functions faced by firm H and L are respectively

$$\begin{aligned} D_H &= \frac{1}{2} + \frac{1 - 2\theta_H}{1+u} \\ D_L &= \frac{2}{1+u}\left(\theta_H - \theta_L\right) \end{aligned}$$

By substituting the definitions of θ_H and θ_L into the demand functions, we can write the profit functions

$$\Pi_H\left(p_H, p_L, s_H, s_L\right) = p_H\left(\frac{1}{2} + \frac{1}{1+u} - \frac{2}{1+u}\frac{p_H - p_L}{s_H - s_L}\right) \tag{3}$$

$$\Pi_L\left(p_H, p_L, s_H, s_L\right) = p_L\frac{2}{1+u}\left(\frac{p_H - p_L}{s_H - s_L} - \frac{p_L}{s_L}\right) \tag{4}$$

maximization of which with respect to p_H and p_L respectively, yields the following reaction functions of the price game:

$$\begin{aligned} p_H(p_L; s_H, s_L) &= \frac{3+u}{8}(s_H - s_L) + \frac{1}{2}p_L \\ p_L(p_H; s_H, s_L) &= \frac{1}{2}\frac{s_L}{s_H}p_H \end{aligned}$$

Therefore, the Nash equilibrium in prices is

$$p_H(s_H, s_L) = \frac{1}{2}s_H \frac{(3+u)}{4s_H - s_L}(s_H - s_L) \tag{5}$$

$$p_L(s_H, s_L) = \frac{1}{4}s_L \frac{(3+u)}{4s_H - s_L}(s_H - s_L) \tag{6}$$

By substituting (5) and (6) into (3) and (4) we obtain the profit functions of the quality stage of the game:

$$\begin{aligned} \Pi_H(s_H, s_L) &= \frac{1}{2}s_H^2 \frac{(3+u)^2 (s_H - s_L)}{(4s_H - s_L)^2 (1+u)} \\ \Pi_L(s_H, s_L) &= \frac{1}{8}s_H s_L \frac{(3+u)^2 (s_H - s_L)}{(4s_H - s_L)^2 (1+u)} \end{aligned}$$

Since Π_H is always increasing in s_H (as should be expected as quality increases are costless), we may conclude that firm H has a dominant strategy in choosing the highest available quality:

$$s_{H1} = \overline{s}$$

Moreover, maximization of Π_L with respect to s_L yields the following optimal quality for firm L:[4]

$$s_{L1}(u) = \frac{4}{7}s_{H1} = \frac{4}{7}\overline{s}$$

so that the solution for prices can be rewritten as:

$$\begin{aligned} p_{H1}(u) &= \frac{(3+u)}{16}\overline{s} \\ p_{L1}(u) &= \frac{(3+u)}{56}\overline{s} \end{aligned}$$

[4]Notice that if we assume that the income of the indiffferent consumers, θ_H and θ_L lie in the modal area, we obtain for all values of u delivering such a solution the same (invariant) quality ratio which characterizes the uniform distribution case (Choi and Shin, 1992).

Finally, we notice that these values of prices and qualities imply the following values for θ_H and θ_L:

$$\theta_{H1}(u) = \frac{5}{16} + \frac{5u}{48} \tag{7}$$

$$\theta_{L1}(u) = \frac{3}{32} + \frac{u}{32} \tag{8}$$

3.2 Case (b): both marginal consumers in the left-tail area

Assume now that $\theta_L, \theta_H \in A$. Then the demand functions faced by the two firms are the following

$$D_H = 1 - \frac{2\theta_H^2}{1-u^2}$$

$$D_L = \frac{2\left(\theta_H^2 - \theta_L^2\right)}{1-u^2}$$

In order to obtain an explicit solution for this case, it is useful to write profits as functions of θ_H and θ_L:

$$\Pi_H(p_H, p_L, s_H, s_L) = p_H D_H = (\theta_H \Delta_s + s_L \theta_L)\left(1 - \frac{2\theta_H^2}{1-u^2}\right) \tag{9}$$

$$\Pi_L(p_H, p_L, s_H, s_L) = p_L D_L = 2 s_L \theta_L \frac{\left(\theta_H^2 - \theta_L^2\right)}{1-u^2} \tag{10}$$

where $\Delta_s = s_H - s_L$ and we make use of the definitions of θ_H and θ_L. Since $\partial\Pi_H/\partial p_H = (\partial\Pi_H/\partial\theta_H)(\partial\theta_H/\partial p_H)$, and $(\partial\theta_H/\partial p_H) \neq 0$, the first order condition for profit maximization of firm H at the price stage can be written as[5]

$$\Delta_s - \frac{6\Delta_s\theta_H^2 - 4\theta_H\theta_L s_L}{1-u^2} = 0$$

Similarly, for firm L, $\partial\Pi_L/\partial p_L = (\partial\Pi_L/\partial\theta_L)(\partial\theta_L/\partial p_L) + (\partial\Pi_L/\partial\theta_H)(\partial\theta_H/\partial p_L)$, so that profit maximization requires[6]

$$\theta_H^2 - 3\theta_L^2 - 2\theta_L\theta_H \frac{s_L}{\Delta_s} = 0$$

[5] The second order conditions for a maximum are verified since Π_H is always concave in θ_H.

[6] The second order conditions require $\frac{d^2\Pi_L}{d\theta_L^2} = 4s_L \frac{(s_L^2 - 3\Delta_s^2)\theta_L - 2s_L\theta_H\Delta_s}{(1-u^2)\Delta_s^2} < 0$. When evaluated at the optimal value $\theta_L = \frac{\theta_H}{3\Delta_s}\left(\sqrt{s_L^2 + 3\Delta_s^2} - s_L\right)$, the numerator of the above expression becomes $= -\frac{\theta_H}{3\Delta_s}\left(\left(\sqrt{s_L^2 + 3\Delta_s^2} - s_L\right)s_L + 3\Delta_s^2\right)\sqrt{s_L^2 + 3\Delta_s^2}$ which is negative for any $s_H > s_L > 0$.

The above reaction functions yield the following equilibrium values of p_H and p_L:

$$\begin{aligned} p_H(s_H, s_L) &= \theta_H \Delta_s + s_L \theta_L = \frac{(3\Delta_s^2 + k)}{\sqrt{6}} \sqrt{\frac{1-u^2}{9\Delta_s^2 + 2k}} && (11) \\ p_L(s_H, s_L) &= s_L \theta_L = \frac{k}{\sqrt{6}} \sqrt{\frac{1-u^2}{9\Delta_s^2 + 2k}} && (12) \end{aligned}$$

where $k = s_L \left(\sqrt{s_L^2 + 3\Delta_s^2} - s_L \right)$

By substituting (11) and (12) into the profit functions (9) and (10), we get

$$\begin{aligned} \Pi_H(s_H, s_L) &= \frac{\sqrt{6}}{3} \sqrt{1-u^2} \frac{(3\Delta_s^2 + k)^2}{\left(\sqrt{9\Delta_s^2 + 2k}\right)^3} \\ \Pi_L(s_H, s_L) &= \frac{\sqrt{6}}{9} \sqrt{1-u^2} \frac{k(3\Delta_s^2 + k)}{\left(\sqrt{9\Delta_s^2 + 2k}\right)^3} \end{aligned}$$

It can be checked that (not surprisingly) also in this case firm H's profits are always increasing in s_H, so that H chooses the maximum available quality s,

$$s_{H2} = \overline{s}$$

while firm L maximizes its profits by setting:[7]

$$s_{L2}(u) = 0.49545\overline{s}$$

Therefore, evaluated at the optimal quality the prices of the two firms are

$$\begin{aligned} p_{H2}(u) &= 0.24806\overline{s}\sqrt{1-u^2} \\ p_{L2}(u) &= 0.061592\overline{s}\sqrt{1-u^2} \end{aligned}$$

As a result, the income indices of the indifferent consumers associated to this solution are:

$$\begin{aligned} \theta_{H2}(u) &= 0.36957\sqrt{(1-u^2)} && (13) \\ \theta_{L2}(u) &= 0.12432\sqrt{(1-u^2)} && (14) \end{aligned}$$

[7] Maximization of $\Pi_L(s_L, s_H)$ yields a solution of the general form $s_L = as_H$, where the constant a is one of the roots of a polynomial of degree 5. Therefore the problem collapses to the maximization of $\Pi_L(as_H, s_H)$ with respect to a. By performing this calculation we obtain the solution in text. This calculation procedure is analogous to that in Motta (1993). Notice also that the solution verifies $a < 1$ and that the second order conditions for a maximum are satisfied for all economically relevant values of a.

3.3 Case (c): θ_L in the left-tail area, θ_H in the modal area

If we now constrain our solution to be such that $\theta_L \in A$, $\theta_H \in B$, then the shape of the demand functions is the following

$$\begin{aligned} D_H &= \frac{1}{2} + \frac{1-2\theta_H}{1+u} \\ D_L &= \frac{1}{2} - \frac{1-2\theta_H}{1+u} - \frac{2\theta_L^2}{1-u^2} \end{aligned}$$

At the price stage of the game, the profit and reaction functions of firm H are the same as those of case (a):

$$\Pi_H(p_H, p_L, s_H, s_L) = p_H\left(\frac{1}{2} + \frac{1}{1+u} - \frac{2}{1+u}\frac{p_H - p_L}{\Delta_s}\right) \tag{15}$$

$$p_H(p_L; s_H, s_L) = \frac{3+u}{8}\Delta_s + \frac{1}{2}p_L \tag{16}$$

As to firm L, by maximizing $\Pi_L(p_H, p_L, s_H, s_L) = p_L D_L$ with respect to p_L, we obtain the following reaction function[8]

$$p_L(p_H; s_H, s_L) = \frac{s_L\sqrt{(1-u)^2(2s_L^2 - 3\Delta_s^2) + 12(1-u)\Delta_s p_H} - 2s_L^2(1-u)}{6\Delta_s} \tag{17}$$

The simultaneous solution of (16) and (17) yields the Nash equilibrium prices:

$$\begin{aligned} p_H(s_H, s_L) = {} & \frac{3+u}{8}\Delta_s + \\ & + \frac{s_L}{24\Delta_s}[\sqrt{3(1-u)}\sqrt{3s_L^2(1-u) + 2\Delta_s^2(1+3u)} - 3s_L(1-u)] \end{aligned} \tag{18}$$

$$p_L(s_H, s_L) = \frac{s_L}{12\Delta_s}[\sqrt{3(1-u)}\sqrt{3s_L^2(1-u) + 2\Delta_s^2(1+3u)} - 3s_L(1-u)] \tag{19}$$

In order to solve the quality stage of the game, we rewrite profits of firm H by substituting (16) into (15):

$$\Pi_H(p_H(p_L, s_H, s_L), p_L, s_H, s_L) = \frac{1}{32}\frac{(\Delta_s(3+u) + 4p_L)^2}{(1+u)\Delta_s} \tag{20}$$

[8] It can be checked that the profit function Π_L is concave in p_L for $\Delta_s > 0$.

Taking into account that at the price stage equilibrium p_L depends on qualities (eqt.19), $d\Pi_H/ds_H = d\Pi_H/d\Delta_s = \partial\Pi_H/\partial\Delta_s + (\partial\Pi_H/\partial p_L)(\partial p_L/\partial\Delta_s)$. According to (20), $\partial\Pi_H/\partial\Delta_s > 0$ provided that $p_H > p_L$, and $\partial\Pi_H/\partial p_L > 0$; on the other hand, according to (19) $\partial p_L/\partial\Delta_s > 0$. Therefore we may conclude that also in case (c) firm H has the incentive to set the maximum quality:

$$s_{H3} = \overline{s}$$

On the other hand, using (19) profits of firm L can be written as

$$\Pi_L = \frac{1}{72}\frac{(1+3u)}{(1+u)}\frac{s_L}{\Delta_s}\left(\sqrt{3(1-u)}\sqrt{3s_L^2(1-u)+2\Delta_s^2(1+3u)} - 3s_L(1-u)\right)$$

Through tedious calculation we obtain that the optimal choice of the low quality s_L is given by

$$s_{L3} = \rho(u)\,\overline{s}$$

where $\rho(u)$ is the following (rather complicated) function:[9]

$$\rho(u) = \frac{1}{2}\left(\frac{9u+7}{3u+5} + z - \sqrt{\frac{2\alpha\sqrt[3]{ab}z - 2^{\frac{2}{3}}(ab)^{\frac{2}{3}}z - \beta z - \gamma\sqrt[3]{ab}}{\sqrt{b}\sqrt[3]{ab}z}}\right)$$

[9] Indeed, the first order conditions for profit maximization are satisfied for $s_L = \rho s_H$, where ρ is a root of the following polynomial of degree 4:

$$(3u+5)Z^4 - (18u+14)Z^3 + (36u+12)Z^2 - (24u+8)Z + 6u + 2$$

where, contrary to the previous cases, the coefficients are a function of the concentration parameter. By setting $u = t+1$ and $Z = x+1$, the above polynomial can be written as

$$3tx^4 - 6tx^3 + 6tx + 3t + 8x^4$$

Formal analysis of the latter shows that it has two complex roots and two distinct real roots for $-\frac{128}{15} \leq t \leq 0$, and therefore for all relevant values of u. By choosing the real root such that $\rho(u) < 1$ (i.e. $s_L < s_H$) we obtain the solution in text. As to the second order conditions, it can be checked that they are verified by substituting specific values of u in the second order derivative and evaluating it at the optimal s_L.

with

$$
\begin{aligned}
a &= \left(\begin{array}{c} 9\left(3u^3-5u^2+u+1\right)+ \\ +\sqrt{-135u^6-702u^5+2223u^4-1220u^3-633u^2+354u+113} \end{array} \right) \\
b &= \left(27u^3+135u^2+225u+125\right)^2 \\
\alpha &= 9\left(3u^3-u^2-7u+5\right) \\
\beta &= 4\sqrt[3]{2}\left(243u^6+1458u^5+2889u^4+1260u^3-2475u^2-2750u-625\right) \\
\gamma &= 2\left(27u^3+207u^2-111u-123\right) \\
z &= \sqrt{\frac{\alpha\sqrt[3]{ab}+2^{\frac{2}{3}}\left(ab\right)^{\frac{2}{3}}+\beta}{\sqrt[3]{ab}\sqrt{b}}}
\end{aligned}
$$

Therefore, at the subgame perfect equilibrium, the prices set by the two firms are:

$$
\begin{aligned}
p_{L3}\left(u\right) &= \frac{1}{12}\frac{\rho\left(u\right)}{\left(1-\rho\left(u\right)\right)}\overline{s}\delta\left(u\right) \\
p_{H3}\left(u\right) &= \frac{3+u}{8}\overline{s}\left(1-\rho\left(u\right)\right)+\frac{1}{24}\frac{\rho\left(u\right)}{\left(1-\rho\left(u\right)\right)}\overline{s}\delta\left(u\right)
\end{aligned}
$$

where

$$
\delta\left(u\right)=\sqrt{3\left(1-u\right)}\sqrt{3\left(\rho\left(u\right)\right)^2\left(1-u\right)+2\left(1-\rho\left(u\right)\right)^2\left(1+3u\right)}-3\rho\left(u\right)\left(1-u\right)
$$

so that the indifferent consumers are identified by the following income indeces:

$$
\theta_{L3}\left(u\right) = \frac{1}{12}\frac{1}{\left(1-\rho\left(u\right)\right)}\delta\left(u\right) \tag{21}
$$

$$
\theta_{H3}\left(u\right) = \frac{3+u}{8}-\frac{1}{24}\frac{\rho\left(u\right)}{\left(1-\rho\left(u\right)\right)^2}\delta\left(u\right) \tag{22}
$$

We are now ready to use our solutions of cases (a) to (c) to discuss the perfect equilibrium of the game as a function of the dispersion parameter u.

3.4 Perfect equilibrium

In the previous section we have solved for prices and qualities by constraining stategies to generate alternative positions of the indifferent consumers. In order to fully parametrize the solution on the income concentration parameter, we now verify which of these solutions actually applies for each value of u. This procedure requires two steps. First of all, for each solution we identify the range of values of u such that such solution is internally consistent (i.e., it satisfies the assumptions under which it is derived). Second, in case there were values of u delivering more than one internally consistent solution, we verify which constrained solution satisfies the Nash equilibrium properties, when the constraint on the position of the marginal consumers is relaxed.

The solution of case (a) holds provided that $\theta_H(u)$ and $\theta_L(u)$ lie in the modal area. Therefore, using (7) and (8) this constrained solution is consistent provided that

$$\theta_{L1}(u) = \frac{3}{32} + \frac{u}{32} \geq \frac{1-u}{2}$$

which implies

$$u \in \left[\frac{13}{17}, 1\right].$$

The solution of case (b) has been derived under the hypothesis that $\theta_H(u)$ and $\theta_L(u)$ lie in the left tail area. Using (13) and (14), consistency is ensured by the condition

$$\theta_{H2}(u) = 0.36958\sqrt{(1-u^2)} \leq \frac{1-u}{2}$$

which defines the following range of values of u:

$$u \in [0, 0.29336]$$

Finally, case (c) offers a solution which holds when $\theta_H(u)$ lies in the modal area and $\theta_L(u)$ lies in the left tail area. On the basis of (21) and (22) the following inqualities must be satisfied:

$$\begin{aligned}
\theta_{L3}(u) &= \frac{1}{12}\frac{1}{(1-\rho(u))}\delta(u) \leq \frac{1-u}{2} \\
\theta_{H3}(u) &= \frac{3+u}{8} - \frac{1}{24}\frac{\rho(u)}{(1-\rho(u))^2}\delta(u) \geq \frac{1-u}{2}
\end{aligned}$$

which are verified for

$$u \in [0.28712, 0.77124]$$

One may notice that there are very small intervals of u such that more than one solution turns out to be consistent. This happens for those values of u such that one of the two indifferent consumers gets close to the kink of the distribution. In particular for $u \in \left[\frac{13}{17}, 0.77124\right]$ it is possible to define consistently both a type (a) and a type (c) constrained solution, while for $u \in [0.28712, 0.29336]$ both a type (b) and a type (c) constrained solution can apply.

For all values of u which deliver only one internally consistent solution, this constrained solution is also the solution of the game with no constraints in the space of strategies. However, when two constrained solutions present themselves, each of them must be checked against profitable deviations involving an enlargment of the strategy space beyond that defined by the conjectured positions of the marginal consumer.

Consider first $u \in \left[\frac{13}{17}, 0.77124\right]$ and case (a) solution. For it to be the unconstrained Nash equilibrium, it should not be profitable for firms to choose an alternative strategy which would deliver a case (c) internally consistent solution. Since firm H always chooses the maximum quality, it is enough to focus on firm L. We have to compare firm L's profits under case (a) solution, and the profits it could obtain if it set an alternative quality such that the case (c) equilibrium prices of L and H evaluated at that quality moved θ_L in the left tail area. The best of these alternative strategies is case (c) optimal quality s_{L3}. Therefore the above comparison boils down to a direct comparison of maximum profits in cases (a) and (c). It can be checked numerically that for $u \in \left[\frac{13}{17}, 0.76779\right)$ profits of firm L are higher under case (c) solution, while the opposite holds for $u \in (0.76779, 0.77124]$. A similar reasoning allows to show that for $u \in [0.28712, 0.29336]$ the case (b) solution applies up to $u = 0.29033$, while case (c) solution holds from that value onwards.[10]

4 Price, quality and income concentration

The analysis developed in the previous section allows to identify the solution of the model for all values of the concentration index u. The pattern of some relevant endogenous variables is shown in Figure 2, drawn under the assumption that the maximum quality $\overline{s}$ be equal to 1.

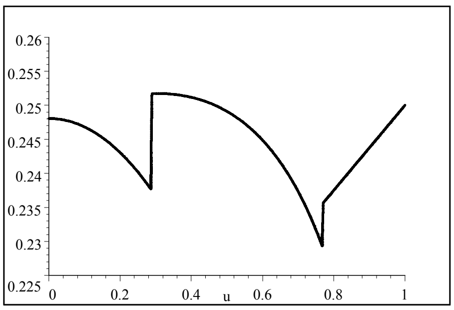

Figure 2a: $p_H(u)$.

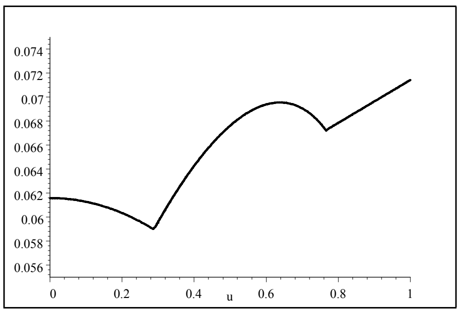

Figure 2b: $p_L(u)$

[10] Clearly, for $u = 0.76779$ and $u = 0.29033$ the model exhibits multiplicity of equilibria.

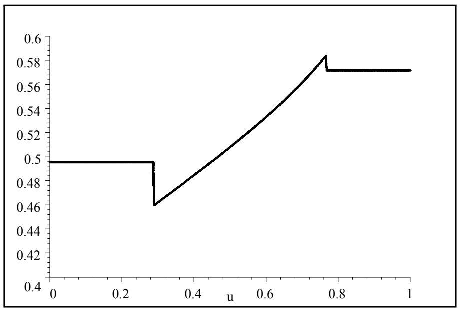

Figure 2.c: $s_L(u)$

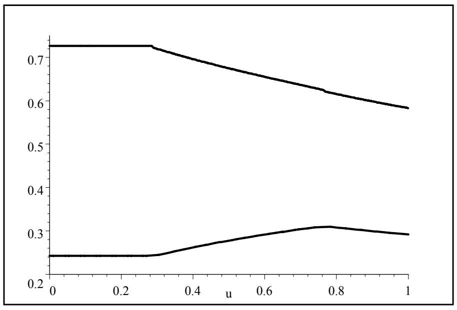

Figure 2.d: Sales of L and H

It is immediately apparent that for many variables the overall pattern is not monotone. On the one hand, the qualitative behaviour of prices, low quality and equilibrium sales differ across the intervals of u defined above; on the other hand, there are variables which behave non monotonically even within those intervals. This suggests that changes in income concentration have a different impact on market equilibrium depending on initial conditions, i.e. depending on whether they affect a more or less dispersed initial distribution. Accordingly, in order to understand the economics behind the pattern of endogenous variables, it is useful to analyse the three intervals separately.

4.1 High dispersion at the initial conditions

This is the situation in which case (a) solution is the perfect Nash equilibrium, holding for $u \in [0.76779, 1]$, according to the above discussion. It is clearly a rather simple case. Optimal qualities are both unaffected by u, while both p_{H1} and p_{L1} (and hedonic prices) are increasing in u, so that they decrease as incomes become more concentrated. Moreover, concentration implies an increase in equilibrium sales (market coverage), and a slight increase in profits for both firms.

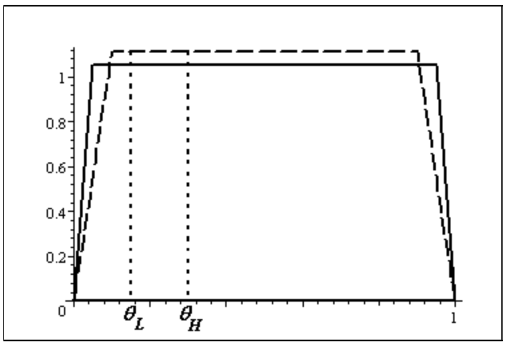

Fig.3a: Income concentration;case (a)

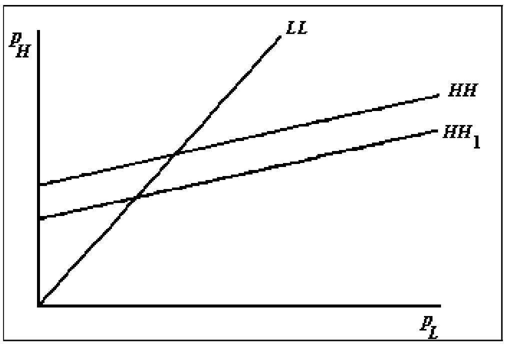

Fig.3b: Reaction functions:price stage

Figure 3 helps us to capture the intuition underlying these results. In Figure 3a two trapezoids are drawn, the solid one being the initial distribution. With θ_H and θ_L we denote the indifferent consumers at the initial conditions. The figure makes clear what happens when the distribution becomes more concentrated: at the initial equilibrium the demand faced by firm H increases, since all previous customers still patronize that firm, while there is an inflow of new customers belonging to the pool of consumers which move from the left tail area to the modal area. From this pool new customers arise also for firm L, adding to the previous ones. For firm H the increase in demand is accompanied by an increase in demand elasticity which prompts a decrease of p_H. Elasticity of demand for firm L, which is unchanged at the initial equilibrium, increases as p_H decreases. In turn, the reduction of p_L increases the elasticity of H's demand, thus stimulating a further reduction in its price.

This process is mirrored in the behaviour of the (positively sloped) reaction functions of the price stage, represented in Figure 3b. Firms H's reaction function HH shifts downwards to HH_1, due to the increase in demand elasticity. The reaction function of L has two properties which deserve attention. First, it does not depend on u, and this reflects the insensitivity of L's demand elasticity to distributional shocks. Second, L's reaction function, LL, is a ray from the origin. Both these properties are due to the density function being a constant within the limits of integrations θ_L and θ_H, and therefore on D_L being homogenous of degree 1 in prices. Given this shape of the reaction function of L, at the new equilibrium in prices, the price ratio p_H/p_L is unchanged – which amounts to saying that prices depend on u only through a (common) multiplicative factor.

This brings us to the analysis of the optimal qualities. In this, as in all the other cases, the choice by H of the maximum quality depends on quality being costless. As for firm L, the homogeneity in prices of its demand function together with the associated properties of optimal prices, imply that u enters the profit function at the quality stage only through a multiplicative factor. The optimal choice of s_L is thus independent of u.[11]

4.2 Low dispersion at the initial conditions

When at the initial conditions incomes are highly concentrated and the Nash equilibrium is given by case (b) solution, $u \in [0, 0.29033]$, our model still delivers optimal qualities which are insensitive to changes in income concen-

[11] Alternatively, one could say that the elasticity with respect to s_L of both the optimal price of L and its demand at the quality stage are independent of u.

tration; however, a negative relation arises between optimal prices and u. As incomes concentrate towards the middle, both p_H and p_L increase. The equilibrium sales of the two firms are unchanged and therefore their profits increase.

We can interpret these results along the same lines as above. Figure 4a shows clearly that the impact effect of an increase in concentration is an increase in the demand for H and a decrease in the demand for L: since the initial distribution is in itself concentrated, a further decrease in u generates a small inflow of new consumers in the market patronizing the L firm, and a more relevant outflow of consumers from the L to the H firm. Contrary to the previous case, for firm H the increase in demand is associated to a decrease in demand elasticity and this prompts, *ceteris paribus*, an increase in p_H. The positively sloped reaction function of H of the price stage of the game shifts upwards in HH_1 (Figure 4b).[12]

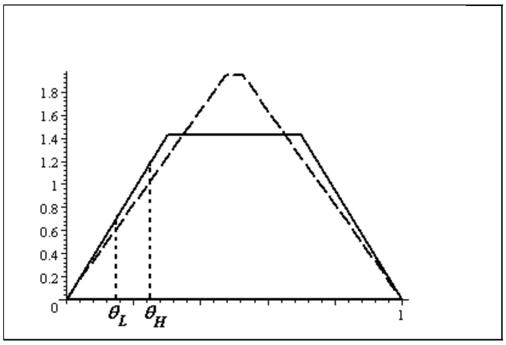

Fig.4a: Income concentration;case (b)

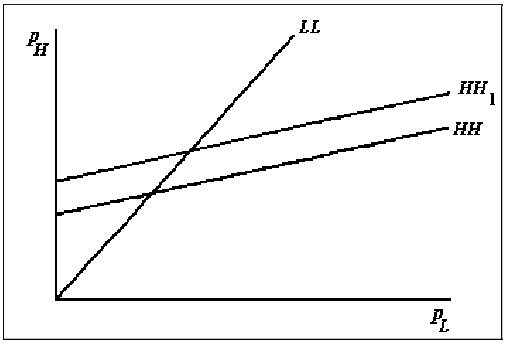

Fig.4b: Reaction functions:price stage

As far as demand for L is concerned, since the density function is linear within the limits of integrations θ_L and θ_H, it is homogenous of degree 2 in prices. This in turn implies that its elasticity is insensitive to u, and that the reaction function of L at the price stage is a again a ray through the origin. The new equilibrium is characterized by higher prices for both firms, due to strategic complementarity, with an unchanged price ratio.

The quality stage of the game has exactly the same properties of case (a).

[12] Notice that the equations of the reaction functions of the H and L firm are respectively:

$$\begin{aligned} p_H &= \frac{2}{3}p_L + \frac{1}{6}\sqrt{\left(4p_L^2 + 6\left(1-u^2\right)\left(s_H - s_L\right)^2\right)} \\ p_L &= \frac{1}{3}\left(2s_L - \sqrt{\left(4s_L^2 + 3s_H^2 - 6s_L s_H\right)}\right) p_H \frac{s_L}{s_H\left(2s_L - s_H\right)} \end{aligned}$$

4.3 Intermediate dispersion at the initial conditions

The analysis of the previous two situations was made easier by the invariance of qualities with respect to u (within its given intervals). This obviously depended on the specific properties of our trapezoid distribution, which in those cases conferred desirable homogeneity properties to the demand faced by firm L. Invariance of qualities means that changes in u do not create any incentive for firm L to alter the intensity of price competition through a different quality strategy.

For intermediate values of u, $u \in [0.29033, 0.76779]$, a situation arises in which, as u decreases, the low quality firm perceives a strong incentive to soften a potential price competition by the high quality one, through a reduction of quality, i.e. a stronger product differentiation.

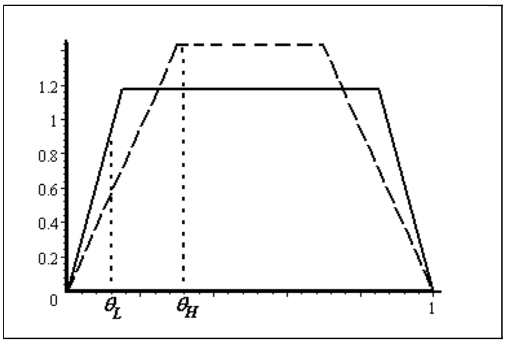

Fig.5a: Income concentration; case (c)

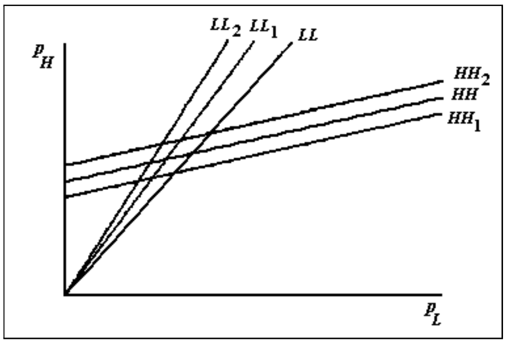

Fig.5b: Reaction functions:price stage

The impact effect of an increase in income concentration in this case is represented in Figure 5a. As in case (a), the demand and the demand elasticity of H both increase. By contrast, the demand faced by L and its elasticity move in opposite directions. In particular, when u is close to the upper bound of the interval, demand increases and its elasticity decreases; the opposite holds for u close to the lower bound. The non-monotone behaviour of D_L is intuitive: when dispersion is still high a significant inflow of new consumers more than outweights the outflow towards the high quality firm; the opposite holds when the initial conditions are close to those of case (b). The non monotone behaviour of price-elasticity depends on the twofold marginal impact of a change of p_L, at the extremes of L's market share: demand becomes more elastic at the richest extreme and less elastic at the poorest extreme. The overall impact of a decrease in u depends on the balancing of these two effects – demand becoming more inelastic (elastic) when L's demand increases (decreases).

According to the above, as u decreases, at unchanged qualities the optimal price of H for given p_L decreases; while, for given p_H, L may increase or decrease its price. This is represented in Figure 5b, which shows the reaction functions of the price stage at the initial qualities. The reaction function of H shifts downwards from HH to HH_1; we assume that the initial conditions are such that the reaction function of L shifts leftwards from LL to LL_1. Prices decrease for both firms.

In this case, however, at the quality stage of the game firm L may improve its performance by softening price competition. Its reaction function in qualities is a ray from the origin, the slope of which is an increasing function of u. Therefore, incomes concentration induces a stronger product differentiation. The wider qualitative distance between the two firms induces both to change their prices: given p_L, the optimal price of H increases, and its reaction function shifts upwards to HH_2. Given p_H, firm L compensates the lower quality of its product through a reduction of its price, and its reaction function shifts leftwards to LL_2. At the new perfect equilibrium p_H is higher and p_L is lower than their initial values. Notice that if we had assumed initial conditions such that, at unchanged qualities, the impact of a decrease in u was to induce a rightward initial shift of LL, then the effect of the quality change on p_L might have not been sufficient to ensure that the latter is indeed lower at the new perfect equilibrium. For values of u close to the upper bound of this case, both prices increase at the two-stage equilibrium.

5 Concluding remarks

In this paper we have developed the analytical solution of a model of vertical differentiation with a non-uniform distribution of consumers' incomes and uncovered market. We have parametrized its solution to the value of a concentration parameter, which is a mean preserving spread over a fixed support. To our knowledge, this is the first attempt to solve explicitly a model with these properties – which has been possible through the useful device of modeling income distribution as a trapezoid.

The above discussion has shown that the comparative statics analysis on the effects of income concentration is strongly dependent on initial conditions. However, it is possible to offer some general considerations on the overall effects of this kind of distributional phenomenon on market equilibrium. First of all, the extent of product differentiation *does depend* on the degree of income concentration. Within the framework of uncovered market, the existence of this link has never been pointed out in the literature. Our analysis shows that the standard invariance of quality result depends heavily on limiting distributional shocks to being simple shifts or stretchings of a uniform distribution. More precisely, if we assume that quality is costless, qualities turn out to be independent of the distribution of income only when the income shock preserves the linearity of the density function for all income classes of consumers patronizing firm L – our cases (a) and (b). Overall, more concentrated incomes imply larger product differentiation: the presence of a large share of consumers in middle income classes stimulates a price competition, whose effects are dampened by firm L through an enlargment of the quality spread. Greater consumers' homogeneity creates a more competitive market environment,[13] which fosters an enlargment of the quality differential.[14]

Moreover, our model confirms the intuition that a more concentrated income distribution extends the coverage of the market and favours the high quality firm. On the one hand, moving from the uniform to the triangular distribution, the market coverage increases from 87.5% to 96.9% of the market. On the other hand, the high-quality advantage, measured by the ratio of the firms' profits, increases as we move from the high dispersion to the low dispersion case – it is 'small' and constant within case (a), 'big' and constant within case (b) and strictly increasing in case (c).

Finally, some remarks on the robustness of our results are in order. Leaving aside the peculiarities generated by the linearities (e.g. the invariance of qualities for some intervals of the concentration parameter), it is clear that the patterns described in each of the cases we have analysed may or may not appear under more regular distribution. However, our stylized distribution allows to identify the key economic mechanism at work and their interplay. The crucial element is the impact change in demand size and demand elasticity generated for each firm by the distributional shock. This clearly depends for both firms on initial conditions. Strategic complementarity at the price stage then defines the extent of price competition. The tougher is the latter, the greater is the incentive to increase product differentiation.

[13] It is in this sense that this model confirms the intuition that stronger consumers' homogeneity may favour market competitiveness – see e.g. Benassi, Chirco and Scrimitore (2002) and Benassi, Cellini and Chirco (2002).

[14] This result is consistent with that obtained by Wauthy (1996), who suggests a negative relationship between income dispersion and the degree of product differentiation, by extending the support of a uniform distribution under full market coverage.

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Appendix

Proof of Proposition 1

Consider firm H's First Order Condition for maximum profits in the price game:

$$1 - F(\theta_H) - \frac{p_H}{s_H - s_L} f(\theta_H) = 0$$

which can be written as

$$1 - F(\theta_H) = b(\theta_H) \tag{A.1}$$

where $b(\theta_H) = (\theta_H + a)\, f(\theta_H)$ and $a = p_L/(s_H - s_L) > 0$. We are going to show that $a > 0$ implies $\theta_H < 1/2$ when this equation satisfied. Since in equilibrium one must have $\theta_L < \theta_H$ and any perfect equilbrium implies a Nash equilibrium in prices at which (A.1) holds, the result follows. We treat a as a positive constant, and letting primes denote derivatives observe the following:

(1) the LHS is monotonically decreasing from 1 to zero.

(2) the RHS is increasing so long as $f'(\theta_H) \geq 0$, since $b'(\theta_H) = f(\theta_H) + (\theta_H + a)\, f'(\theta_H)$; notice that $b(\cdot)$, a continuous function, is such that $b(1) = b(0) = 0$: hence it has at least a maximum at some $\widehat{\theta}_H$. The latter has to satisfy $\widehat{\theta}_H > 1/2$, as by construction $b'(\theta_H) = f(\theta_H) + (\theta_H + a)\, f'(\theta_H) > 0$ for $f'(\theta_H) \geq 0$, which is certainly true for $\theta_H \leq 1/2$.

(3) $b(1/2) = (a + 1/2) f(1/2) > 1/2 = 1 - F(1/2)$.

All of which gives the following. By (3), $b(1/2) > 1 - F(1/2)$; but since $b(0) = af(0) = 0 < 1 - F(0) = 1$, there is one $\theta_H^* < 1/2$ such that equation (A.1) holds; due to (1) and (2), θ_H^* is unique over $[0, \widehat{\theta}_H]$, with the second order conditions for firm H implying that $1 - F(\theta_H)$ cuts b from above (hence, its derivative is less than that of b at θ_H^*). For $\theta_H > \widehat{\theta}_H > 1/2$, $1 - F(\theta_H)$ cannot cross again $b(\theta_H)$ from above. Indeed, suppose that such a point, θ_H' say, exists: that would imply a further crossing at a point $\overset{\circ}{\theta}_H \in (\widehat{\theta}_H, \theta_H')$ such that

$$\begin{aligned} -2f(\overset{\circ}{\theta}_H) - (a + \overset{\circ}{\theta}_H) f'(\overset{\circ}{\theta}_H) &> 0 \qquad \text{(A.2a)} \\ -2f(\theta_H') - (a + \theta_H') f'(\theta_H') &< 0 \qquad \text{(A.2b)} \end{aligned}$$

that is, $A = 2[f(\theta_H') - f(\overset{\circ}{\theta}_H)] + a[f'(\theta_H') - f'(\overset{\circ}{\theta}_H)] + [\theta_H' f'(\theta_H') - \overset{\circ}{\theta}_H f'(\overset{\circ}{\theta}_H)] > 0$, which cannot be true since $1/2 < \overset{\circ}{\theta}_H < \theta_H'$ implies $f(\overset{\circ}{\theta}_H) \geq f(\theta_H')$ and $0 \geq f'(\overset{\circ}{\theta}_H) \geq f'(\theta_H')$ so that in fact $A < 0$ and we have a contradiction: if (A.2a) holds, (A.2b) cannot hold.■